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1973 J. Phys. A: Math. Nucl. Gen. 6 826

(http://iopscience.iop.org/0301-0015/6/6/011)

J. Phys. A: Math., Nucl. Gen., Vol. 6, June 1973 Printed in Great Britain 

# Approach to equilibrium of single mode radiation in a cavity

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Received 26 June 1972, in revised form 20 November 1972

**Abstract.** The approach to equilibrium of single mode radiation in a cavity is studied. The master equation formed from the diagonal elements of the density matrix is solved in general for arbitrary initial photon probability distributions. Several examples are studied including the decay of an initial Poisson distribution (coherent state) to the equilibrium Bose–Einstein distribution. The concept of a mixed Poisson process is introduced and its physical implications examined in the context of the present problem. A general expression for the nonstationary correlation function of the photon field is also obtained.

## 1. Introduction

The purpose of this paper is to examine the approach to equilibrium of the photon probability distribution of a single mode radiation field in a cavity (maintained at constant temperature) given an initial distribution of photons. If we work in the number representation of the photon field, then only the diagonal elements of the photon density matrix will enter into the analysis through a master equation for the time-dependent probability distribution. Shimoda *et al* (1957) have already examined some aspects of this problem in their pioneering paper on the (linear) photon description of the maser. Whereas they were interested in *amplifying* solutions, we seek *decaying* solutions.

We have obtained an explicit evaluation of the first and second order probability distributions of the photon field; furthermore, we have obtained a *general expression* (in closed form) for the *nonstationary correlation function* of the photon field. Several examples of initial distributions are considered, particularly an initial Poisson distribution (coherent state in the number representation).

We also introduce the concept of a mixed Poisson process and examine its physical implications in the context of the present problem. It is shown that after a sufficient length of time ($t > 0$), the number of photons in the cavity becomes and stays a mixed Poisson process. For an initial Poisson distribution (coherent state), the process is mixed Poisson for all time. However, this is not generally true as we will prove.

Our problem is another version of the damped harmonic oscillator problem already considered by Glauber (1970) from the point of view of his coherent state representation.

## 2. Basic equations

Let $n(t)$ be the number of photons in the cavity at time $t$. $n(t)$ is a stochastic process whose associated first-order probability distribution is $P_n(t)$, the probability of having $n$ photons in the cavity at time $t$. Assuming the cavity is maintained at constant temperature, the master equation for $P_n(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}P_n(t) = \alpha(n+1)P_{n+1}(t)+\beta nP_{n-1}(t)-\alpha nP_n(t)-\beta(n+1)P_n(t) \tag{2.1}$$

where $\alpha$ is the absorption coefficient and $\beta$ the emission coefficient. This equation was first derived by Shimoda *et al* (1957) and has since been derived by density matrix techniques (Scully and Lamb 1968, 1969, Scully 1970, Pike 1970). Equation (2.1) is a differential difference equation of a type that occurs in the stochastic theory of birth–death processes (Bailey 1964).

Equation (2.1) is conveniently solved via use of a generating function $Q(\lambda, t)$ defined by

$$Q(\lambda, t) = \sum_{n=0}^{\infty} P_n(t)(1-\lambda)^n. \tag{2.2}$$

If $Q(\lambda, t)$ is known, $P_n(t)$ can be obtained in principle by differentiation

$$P_n(t) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}Q(\lambda, t)\bigg|_{\lambda=1}. \tag{2.3}$$

The generating function can be shown to satisfy the first-order partial differential equation

$$\frac{\partial Q}{\partial t} = \lambda\{(\beta-\alpha)-\beta\lambda\}\frac{\partial Q}{\partial\lambda}-\beta\lambda Q. \tag{2.4}$$

The initial conditions on $Q$ are:

$$Q(0, t) = \sum_{n=0}^{\infty} P_n(t) = 1 \tag{2.5}$$

$$Q(\lambda, 0) = \sum_{n=0}^{\infty} P_n(0)(1-\lambda)^n. \tag{2.6}$$

We will utilize Lagrange's method to solve equation (2.4) rather than employ the usual Laplace transform method because of the simplicity and power of the former method. Reference is made to Bailey (1964) for an excellent account of the method. We can show that the general solution of equation (2.4) can be written as

$$Q(\lambda, t) = \{(\alpha-\beta)+\beta\lambda\}^{-1}g\left(\frac{\lambda}{\alpha-\beta+\beta\lambda}\exp\{-(\alpha-\beta)t\}\right) \tag{2.7}$$

where $g$ is to be determined by the initial condition at $t = 0$ as given in equation (2.6). This is most easily accomplished by a change of variable such that the argument of $g$ is a single parameter. Consequently we set

$$y(\lambda, t) \equiv \left(\frac{\lambda}{\alpha-\beta+\beta\lambda}\right)\exp\{-(\alpha-\beta)t\}$$

and solve for $\lambda$ in terms of $y(\lambda, 0)$. We then write equation (2.7), evaluated at $t = 0$, in terms of $y(\lambda, 0)$; the final result is

$$g(y(\lambda, 0)) = \frac{\alpha-\beta}{1-\beta y(\lambda, 0)}Q\left(\frac{(\alpha-\beta)y(\lambda, 0)}{1-\beta y(\lambda, 0)}, 0\right). \tag{2.8}$$

The function $g(y(\lambda, t))$ is then determined by observing that $y(\lambda, t) = y(\lambda, 0)\exp\{-(\alpha-\beta)t\}$. Consequently, we obtain the general time-dependent behaviour of $Q(\lambda, t)$ by replacing $y(\lambda, 0)$ with $y(\lambda, t)$ in equation (2.8).

Before considering any specific examples of initial conditions, we note time enters the arbitrary function $g$ only in the exponential of its argument. Under ordinary circumstances (ie non-lasing action), $\alpha > \beta$ and therefore at equilibrium ($t = \infty$),

$$Q(\lambda, \infty) = \frac{g(0)}{\alpha-\beta+\beta\lambda} = \frac{\alpha-\beta}{\alpha-\beta+\beta\lambda}. \tag{2.9}$$

The $n$th derivative of $Q(\lambda, \infty)$ is easily evaluated and the resultant photon distribution function is

$$P_n(\infty) = \frac{(\bar{n}(\infty))^n}{(1+\bar{n}(\infty))^{n+1}} \tag{2.10}$$

where $\bar{n}(\infty) = \beta/(\alpha-\beta)$ is the mean number of photons in the cavity at equilibrium. The right hand side of equation (2.10) is the Bose–Einstein distribution expressed in terms of its mean value.

When $\beta > \alpha$ so that lasing action occurs then the solutions of equation (2.1) are unbounded functions of time. However, the *linear* master equation given in equation (2.1) is too simple a model and non-linear terms must be added to it, reference is made to Scully and Lamb (1967) for this aspect of the problem. In what follows we tacitly assume $\alpha > \beta$.

## 3. Mixed Poisson processes

One difficulty with the generating function approach is the necessity of having to take high-order derivatives of $Q$ in order to obtain $P_n$. Since $Q$ is generally a complicated function, taking of high-order derivatives rapidly becomes an unpleasant, if not hopeless, task. Instead, we introduce a new function $W(s, t)$, as the inverse Laplace transform of $Q(\lambda, t)$

$$Q(\lambda, t) = \int_0^\infty W(s, t)\,e^{-\lambda s}\,ds. \tag{3.1}$$

Repeated differentiation of $Q$ with respect to $\lambda$ according to equation (2.4) leads to the result

$$P_n(t) = \int_0^\infty W(s, t)\frac{s^n e^{-s}}{n!}\,ds. \tag{3.2}$$

The usefulness of this device is obvious. Provided we can evaluate $W(s, t)$ given $Q(\lambda, t)$, then $P_n(t)$ can be obtained by integration rather than by repeated differentiation. Even if $W(s, t)$ cannot be evaluated analytically or is so complicated that equation (3.2) defies explicit evaluation, we can still evaluate the integral numerically using Gauss–Laguerre quadrature.

This integral representation of $P_n(t)$ is more than just a useful computational device. Suppose that $W(s, t)$ were itself a probability density function over the interval $(0, \infty)$

with respect to $s$. A necessary and sufficient condition for this to be true is for $W(s, t)$ to be a *real* function and simultaneously satisfy:

$$\int_0^\infty W(s, t)\,ds = 1 \qquad \text{for all } t \tag{3.3}$$

and

$$W(s, t) \geqslant 0 \qquad \text{for all } s \text{ and } t. \tag{3.4}$$

The *first* condition is automatically satisfied by virtue of equations (3.1) and (2.5) irrespective of the second condition. The possible non-negative character of $W(s, t)$ has to be handled on an individual basis.

*If the above conditions are satisfied*, then $P_n(t)$ defines a *mixed Poisson process* (the terms *compound* and *generalized* have also been used). The interpretation is simple. The real variable $s$ in equation (3.2) is non-negative since it is confined to the interval $(0, \infty)$. The factor $s^n \exp(-s)/n!$ in the integrand is a Poisson distribution with respect to $s$, the mean of the distribution. Consequently $P_n(t)$ can be thought of as a distribution formed from a Poisson distribution in which the mean $s$ is a random variable having probability density $W(s, t)$. A detailed summary of many of the pertinent properties of mixed Poisson processes along with an extensive bibliography is given in Haight (1966).

The mere fact that $P_n(t)$ is expressible in the form given in equation (3.2) does not mean that $P_n(t)$ is mixed Poisson, since $W(s, t)$ can be negative. The important point is that $W(s, t)$ simultaneously satisfy equations (3.3) and (3.4). That a number of investigators, particularly in photoelectron counting statistics, have not noticed this distinction has led to some confusion.

When $P_n(t)$ is a mixed Poisson process, then the factorial moment of order $r$ of $P_n(t)$ is equal to the $r$th moment about the origin of the mixing density $W(s, t)$. The mean and variance of $n(t)$ in terms of $W(s, t)$ are:

$$\bar{n}(t) = \bar{s}(t) \tag{3.5}$$

$$\text{var}\, n(t) = \bar{n}(t) + \text{var}\, s(t) \tag{3.6}$$

where

$$\text{var}\, s(t) = \int_0^\infty (s-\bar{s})^2 W(s, t)\,ds. \tag{3.7}$$

The expression for the variance is extremely interesting because it shows that the total variance of the number of photons in the cavity is composed of two terms: the first term is the variance of a Poisson process and would be the only term if the photons showed no tendency to cluster. The second term is the contribution of the mixing density $W(s, t)$. Since this second contribution is positive, we have an important result; namely, *the variance of the mixed Poisson is greater than the variance of a simple Poisson having the same mean.* If the variance of $n(t)$ were less than that of the corresponding simple Poisson, then $n(t)$ could not be a mixed Poisson process even though it possessed the integral form given in equation (3.2). The mixing density contribution to the variance of $n(t)$ is simply a manifestation of the fact that photons have a tendency to cluster.

It is a simple matter to prove that $n(\infty)$ is a mixed Poisson process by direct manipulation of equation (2.10).

## 4. Coherent state initially

If we choose the initial distribution to be Poisson, that is,

$$P_n(0) = \frac{(\bar{n}(0))^n \exp(-\bar{n}(0))}{n!} \tag{4.1}$$

then the initial state is a *coherent state* (Glauber 1963) in the number representation. The coherent state then decays to the equilibrium state and we wish to follow the relaxation process through a study of $P_n(t)$.

The generating function corresponding to equation (4.1) is

$$Q(\lambda, 0) = \exp(-\lambda\bar{n}(0)). \tag{4.2}$$

Following the procedure outlined in § 2, it is straightforward to show that the time-dependent generating function has the form

$$Q(\lambda, t) = \frac{1}{1+a(t)\lambda} \exp\left(\frac{-c(t)\lambda}{1+a(t)\lambda}\right) \tag{4.3}$$

where

$$a(t) \equiv \bar{n}(\infty)(1-e^{-\gamma}), \qquad \gamma = \frac{\beta t}{\bar{n}(\infty)} = \frac{\tau}{\bar{n}(\infty)} \tag{4.4}$$

$$c(t) \equiv \bar{n}(0)\, e^{-\gamma}. \tag{4.5}$$

The mean number of photons is

$$\bar{n}(t) = \bar{n}(\infty) + (\bar{n}(0) - \bar{n}(\infty))\, e^{-\gamma}. \tag{46}$$

We postpone the calculation of the variance until we have the mixing density $W(s, t)$.

In order to calculate $W(s, t)$, let us substitute $Q(\lambda, t)$ as given by equation (4.3) into equation (3.2) and make the change of variable $z = (1+a\lambda)s/a$. The result is

$$W(s,t) = \frac{1}{2\pi i a} \exp\left(-\frac{1}{a}(c+s)\right) \int_{z_0 - i\infty}^{z_0 + i\infty} \frac{1}{z} \exp\left(\frac{cs}{a^2 z} + z\right) dz$$

$$= \frac{1}{a} \exp\left(-\frac{1}{a}(c+s)\right) I_0\left(\frac{2}{a}\sqrt{(cs)}\right), \qquad 0 \leqslant s < \infty. \tag{4.7}$$

The modified Bessel function $I_0(x)$ is positive for $x \geqslant 0$; consequently $W(s, t)$ is always non-negative. *Therefore, the relaxation of an initial coherent state to the equilibrium Bose–Einstein state is a mixed Poisson process for all times.*

To obtain the photon distribution itself, we substitute equation (4.7) into equation (3.2) and make the change of variables

$$z = \frac{s}{a}(1+a), \qquad x = -\frac{c}{a(1+a)}.$$

We have

$$P_n(t) = \left(\frac{a}{1+a}\right)^n \frac{e^{-c/a}}{1+a}\int_0^\infty \frac{z^n}{n!} J_0(a\sqrt{(zx)})\, e^{-z}\, dz$$

$$= \frac{1}{1+a}\left(\frac{a}{1+a}\right)^n \exp\left(-\frac{c}{1+a}\right) L_n\left(\frac{-c}{a(1+a)}\right). \tag{4.8}$$

As $t$ (or $\gamma$) approaches infinity, the argument of the Laguerre polynomial tends to zero (but $L_n(0) = 1$) with the result $P_n(t)$ tends to the Bose–Einstein distribution.

This same distribution appears in the theory of photoelectron counting statistics for the superposition of a coherent and a chaotic excitation (Lachs 1965, Glauber 1966). The time dependence is basically inverse to our situation. In the photon counting case, time corresponds to the time interval the photon counter is exposed to the field. The counting distribution is then Bose–Einstein for very short times and tends to the Poisson as the time interval is increased; just the opposite of our situation.

We calculated the distribution function for some typical values and the results are shown in figures 1 and 2.

Since $W(s, t)$ is now known, it is a straightforward calculation to show that

$$\operatorname{var} n(t) = \bar{n}(t) - \left(\frac{\bar{n}(0)}{\bar{n}(\infty)}(e^\gamma - 1)^{-1}\right)^2 + 2L_2\left(-\frac{\bar{n}(0)}{\bar{n}(\infty)}(e^\gamma - 1)^{-1}\right). \tag{4.9}$$

## 5. *m* photons initially

Let us now consider the case of $m$ photons initially in the cavity with

$$\bar{n}(0) = m, \qquad \operatorname{var} n(0) = 0 \tag{5.1}$$

so that the initial distribution is the unit distribution

$$P_m(0) = 1, \qquad P_n(0) = 0 \qquad \text{for } n \neq m. \tag{5.2}$$

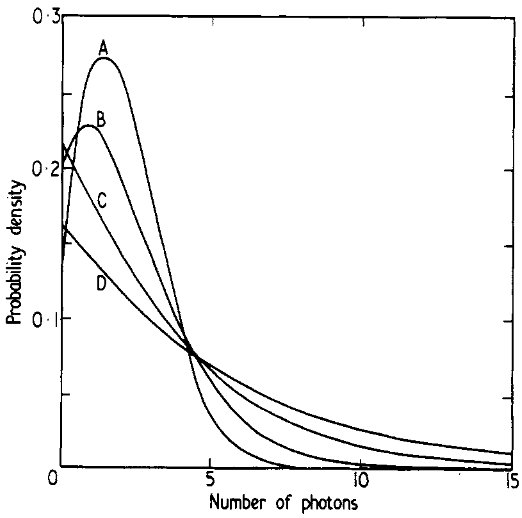


**Figure 1.** $P_n(\tau)$ for an initial coherent state with $\bar{n}(0) = 2$, and $\bar{n}(\infty) = 5$: A $\tau = 0$, B $\tau = 0{\cdot}5$, C $\tau = 2{\cdot}5$ and D $\tau = \infty$.

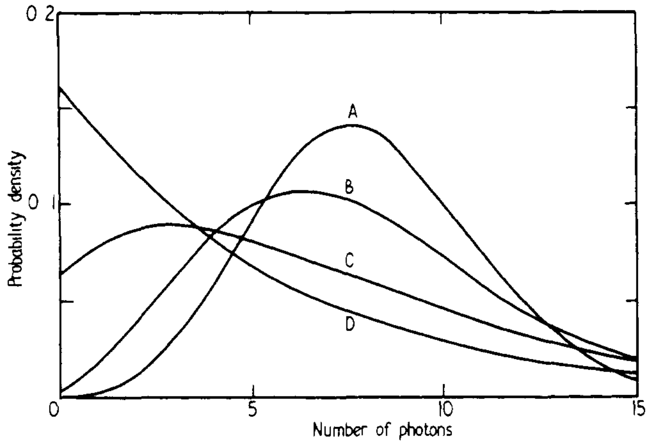


**Figure 2.** $P_n(\tau)$ for an initial coherent state with $\bar{n}(0) = 8$, and $\bar{n}(\infty) = 5$: A $\tau = 0$, B $\tau = 0.5$, C $\tau = 2.5$ and D $\tau = \infty$.

*We will find that the cases $m = 0$ and $m > 0$ possess an entirely different initial (ie small time) behaviour.*

Another interpretation of equation (5.2) is as a *conditional* probability, namely the probability of having $n$ photons in the cavity at time $t$ given that we had $m$ photons at time zero. In the usual notation this is written as $P(n, t|m, 0)$. The importance of this statement lies in the fact that it allows us to obtain the correlation function of the photons (see § 6).

One can easily show that the initial generating function is

$$Q(\lambda, 0) = (1-\lambda)^m. \tag{5.3}$$

The formal procedure for determining $Q(\lambda, t)$ leads to

$$Q(\lambda, t) = \frac{(1-\lambda\, e^{-\gamma} + a(t)\lambda)^m}{(1+a(t)\lambda)^{m+1}} \tag{5.4}$$

where $a(t)$ is defined in equation (4.4).

The mean and variance are:

$$\bar{n}(t) = \bar{n}(\infty) + (m - \bar{n}(\infty))\, e^{-\gamma} \tag{5.5}$$

$$\text{var}\, n(t) = \bar{n}(t) + (\bar{n}(t))^2 - (m+m^2)\, e^{-2\gamma}. \tag{5.6}$$

The expression for the mean is identical with that of the coherent state, equation (4.6), and in fact any initial distribution of photons leads to equation (5.5) as Siegman (1964) has shown. Reference is made to Loudon (1970) for a discussion of the *variance* of an arbitrary initial distribution of photons.

If this process is to be mixed Poisson, then by equation (5.6)

$$(\bar{n}(t))^2 - (m+m^2)\, e^{-2\gamma} \geqslant 0 \tag{5.7}$$

for all times $0 \leqslant t < \infty$. Clearly such a condition cannot be satisfied uniformly in time if $m > 0$ (although it is true for $m = 0$). In the general case of arbitrary $m$ and $\bar{n}(\infty)$, the left hand side of equation (5.7) will be negative for some finite time after which it will become positive due to the dominating effect of the exponential. The var $n(t)$

during the initial stages, is *less* than that of the corresponding simple Poisson distribution; this fact precludes the possibility that relaxation to the Bose–Einstein distribution forms a mixed Poisson process *uniformly* in time. We already know that the long time behaviour of any initial distribution must be mixed Poisson.

In view of the behaviour of the variance, we know that $W(s, t)$ can only be a probability density after an initial passage of time depending on $m$ and $\bar{n}(\infty)$. In the short time region $W(s, t)$ must take on negative values. This is easily verified by noting that $W(s, t)$ has the form (see the appendix):

$$W(s, t) = \frac{e^{-s/a}}{a}\left(\frac{a\,e^{\gamma}-1}{a\,e^{\gamma}}\right)^{m} L_m\left(\frac{-s}{a(a\,e^{\gamma}-1)}\right). \tag{5.8}$$

The condition that $W(s, t) \geqslant 0$ is equivalent to requiring that the argument of the Laguerre polynomial be negative, since $L_m(-x) \geqslant 0$ for all $m$ and $x > 0$. We can always find a time small enough for $(a\,e^{\gamma}-1)$ to be negative, namely $e^{\gamma} < a^{-1}$. The argument of the Laguerre polynomial is then positive and $W(s, t)$ takes on negative values. For longer times, however, just the opposite holds and $W(s, t) \geqslant 0$. Once $W(s, t)$ is positive, it remains positive for all future times and the process is mixed Poisson.

Our primary purpose, in this section, was to demonstrate the unusual initial behaviour of the unit distribution *vis à vis* the mixed Poisson process. However, the formal expression for $P_n(t)$ is not difficult to derive. We simply substitute equation (A.6) into equation (3.2) and employ the integral

$$\int_0^{\infty} e^{-zx}x^k L_l(x)\,dx = k!z^{-k-1}\;{}_2F_1(-l, k+1, 1, z^{-1}). \tag{5.9}$$

Since $l$ is a positive integer, it follows from the theory of hypergeometric functions that ${}_2F_1$ is a polynomial in $z^{-1}$ of degree $l$ and not an infinite series. The final result is

$$P_n(t) = \frac{a^n}{(1+a)^{n+1}}\sum_{l=0}^{m}\binom{m}{l}(-a^{-1}\,e^{-\gamma})^l\;{}_2F_1(-l, n+1, 1, (1+a)^{-1}) \tag{5.10}$$

for $m > 0$ and

$$P_n(t) = \frac{1}{1+\bar{n}(t)}\left(\frac{\bar{n}(t)}{1+\bar{n}(t)}\right)^n \tag{5.11}$$

for $m = 0$. The probability distribution for the case of no photons at $t = 0$ is always a Bose–Einstein distribution whose mean is a function of time given by equation (4.5). The solution given by equation (5.11) is known (Pike 1970).

If the initial condition is taken at $t$, rather than at zero, then we are calculating $P(n, t_2|m, t_1)$ with $t_2 \geqslant t_1$. Since the process $n(t)$ has stationary increments, the only effect is to replace $t$ by $(t_2 - t_1)$ in equation (5.10).

In the limit as $t \to \infty$, equation (5.10) tends to the Bose–Einstein distribution. Thus, we can also interpret the Bose–Einstein distribution as a conditional distribution.

## 6. Correlation function

We now turn our attention to the correlation function $G(t_2, t_1) = G(t_1, t_2)$ which is defined as

$$G(t_2, t_1) = E\{(n(t_2)-\bar{n}(t_2))(n(t_1)-\bar{n}(t_1))\}$$
$$= E(n(t_2)n(t_1))-\bar{n}(t_2)\bar{n}(t_1). \tag{6.1}$$

The first term on the right hand side is by definition

$$E(n(t_2)n(t_1)) = \sum_m \sum_n mnP(n, t_2 ; m, t_1) \tag{6.2}$$

where $P(n, t_2 ; m, t_1)$ is the *joint* probability of obtaining $m$ photons at $t_1$ and $n$ photons at $t_2$; $(t_2 \geqslant t_1)$. We can rewrite this in terms of the *conditional* probability $P(n, t_2|m, t_1)$ already derived in the previous section; thus

$$E(n(t_2)n(t_1)) = \sum_m mP_m(t_1) \sum_n nP(n, t_2|m, t_1)$$
$$= \sum_m mP_m(t_1)\left(-\frac{\partial}{\partial\lambda}Q(\lambda, t_2-t_1)\right)\Bigg|_{\lambda=0}$$
$$= \sum_m mP_m(t_1)\{\bar{n}(\infty)+(m-\bar{n}(\infty))\exp(-|\gamma_2-\gamma_1|)\}. \tag{6.3}$$

This series can be expressed in terms of the mean and variance of $n(t_1)$. Hence,

$$G(t_2, t_1) = \{1-\exp(-|\gamma_2-\gamma_1|)\}\bar{n}(\infty)\bar{n}(t_1)+\exp(-|\gamma_2-\gamma_1|)(n(t_1))^2$$
$$+\exp(-|\gamma_2-\gamma_1|)\operatorname{var}(n(t_1))-\bar{n}(t_2)\bar{n}(t_1). \tag{6.4}$$

This result is *perfectly general* and its explicit evaluation depends on the initial distribution of photons via the mean and variance.

Since the underlying process is nonstationary, the correlation function depends on both $t_2$ and $t_1$. However, as we approach the steady state $G(t_2, t_1)$ must become a function of $(t_2-t_1)$ and not $t_2$, $t_1$ individually. This can be demonstrated by taking any initial distribution and letting $t_2$ and $t_1$ become infinite in such a manner that $(t_2-t_1)$ remains finite.

The limiting form of the correlation function at equilibrium is easily obtained directly via the Bose–Einstein distribution. We have

$$\bar{n}(t_1) = \bar{n}(\infty)$$
$$\operatorname{var}(n(t_1)) = (\bar{n}(\infty))^2+\bar{n}(\infty), \tag{6.5}$$

consequently,

$$G(|t_2-t_1|) = \{(\bar{n}(\infty))^2+\bar{n}(\infty)\}\exp(-|\gamma_2-\gamma_1|) \tag{6.6}$$

is the correlation function at thermal equilibrium.

## Acknowledgments

We wish to thank Professor R Glauber for stimulating discussions on problems in quantum optics. Barakat's research was supported in part by Air Force Office of Scientific Research (AFSC) under Contract F44620-72-C-0063 with Bolt Beranek and Newman Inc.

## Appendix

The analysis required to evaluate $W(s, t)$ from $Q(\lambda, t)$ for the initial condition of § 5 is outlined. To begin we expand $Q(\lambda, t)$, as given in equation (5.2), in terms of a binomial series

$$Q(\lambda, t) = \frac{1}{1+a\lambda}\sum_{l=0}^{m}\binom{m}{l}(-1)^l\left(\frac{\lambda\,\mathrm{e}^{-\gamma}}{1+a\lambda}\right)^l. \tag{A.1}$$

The corresponding expression for $W(s, t)$ is

$$W(s, t) = \sum_{l=0}^{m}\binom{m}{l}(-1)^l\,\mathrm{e}^{-l\gamma}a^{-l-1}W_l(s, t), \tag{A.2}$$

where

$$W_l(s, t) \equiv \frac{1}{2\pi\mathrm{i}}\int_{\lambda_0-\mathrm{i}\infty}^{\lambda_0+\mathrm{i}\infty}\lambda^l\,\mathrm{e}^{s\lambda}(\lambda+a^{-1})^{-l-1}\,\mathrm{d}\lambda. \tag{A.3}$$

This integral can be evaluated by contour integration. Since $s$ is real and non-negative, convergence is assured by closing the contour to the left. Furthermore $a > 0$ is that the pole of order $(l+1)$ at $\lambda = -a^{-1}$ lies on the negative real axis. Upon applying the Cauchy residue theorem, we obtain

$$W_l(s, t) = \frac{1}{l!}\frac{\mathrm{d}^l}{d\lambda^l}(\lambda^l\,\mathrm{e}^{s\lambda})\bigg|_{\lambda=-a^{-1}}. \tag{A.4}$$

The right hand side of this expression is reminiscent of Roderigue's formula for Laguerre polynomials, in fact

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n\,\mathrm{e}^{-x}) = n!\,\mathrm{e}^{-x}L_n(x). \tag{A.5}$$

If we set $x = s/a$, then by obvious manipulations, we obtain $W(s, t)$ in the form

$$W(s, t) = \frac{1}{a}\sum_{l=0}^{m}\binom{m}{l}(-1)^l\,\mathrm{e}^{-s/a}(a^{-1}\,\mathrm{e}^{-\gamma})^l L_l\left(\frac{s}{a}\right). \tag{A.6}$$

Fortunately, this finite series can be summed via the identity

$$L_m(\mu z) = \sum_{l=0}^{m}\binom{m}{l}\mu^l(1-\mu)^{m-l}L_l(z) \tag{A.7}$$

upon setting

$$\mu \equiv (1-a\,\mathrm{e}^{\gamma})^{-1}, \qquad z = s/a. \tag{A.8}$$

The final result is

$$W(s, t) = \frac{\mathrm{e}^{-s/a}}{a}\left(\frac{a\,\mathrm{e}^{\gamma}-1}{a\,\mathrm{e}^{\gamma}}\right)^m L_m\left(\frac{-s}{a(a\,\mathrm{e}^{\gamma}-1)}\right). \tag{A.9}$$

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